

Vortex Pull by an External Current

G.N. Stratopoulos¹

*Department of Physics and Institute of Plasma Physics, University of Crete,
and Research Center of Crete,
P.O.Box 2208, 710 03 Heraklion, Crete; Greece*

Abstract

In the context of a dynamical Ginzburg-Landau model it is shown numerically that under the influence of a homogeneous external current \mathbf{J} the vortex drifts against the current with velocity $\mathbf{V} = -\mathbf{J}$ in agreement to earlier analytical predictions. In the presence of dissipation the vortex undergoes skew deflection at an angle $90^\circ < \delta < 180^\circ$ with respect to the external current. It is shown analytically and verified numerically that the angle δ and the speed of the vortex are linked through a simple mathematical relation.

¹ email: stratos@physics.uch.gr

1 Introduction

In a recent publication [1] it was proposed to study the dynamics of isolated Abrikosov vortices in the framework of a phenomenological effective field theoretic model [2], which alternatively can be viewed as a time dependent version of the Ginzburg-Landau equation (TDGL). Therein, it was possible to derive analytically the equation of motion of the guiding center of the vortex (Hall equation), under the influence of any kind of external force [3]. According to the latter, the guiding center of the vortex moves in a direction perpendicular to the externally applied force in analogy to the planar motion of charged particles in the presence of a perpendicular magnetic field. The importance of the Hall equation stems from theoretical considerations which suggest that the guiding center describes quite accurately the “mean” position of the vortex, a suggestion which was verified and quantified by a recent numerical study [4]. It is thus possible to obtain useful information concerning the motion of a vortex by simply invoking Hall equation. For instance, by plain implementation of the latter one finds that in the presence of a homogeneous external current the vortex drifts against the current.

By construction, the Hall equation is limited to the description of the gross features of the motion of the vortex. On the other hand, an understanding of the finer details of its motion requires a detailed solution of the TDGL equations. Such an undertaking has to rely on numerical methods, due to the nonlinear nature of the relevant equations. The main purpose of this paper is to report on a numerical study of the dynamics of vortices within this model. In particular we will study the response of a vortex to an external current with or without dissipation, an issue of obvious interest in the context of superconductivity [5],[6].

The paper is organized as follows. Section 2 contains a general introduction to the model. Its relevance to the physics of the superconductor is commented upon, and the main theoretical predictions concerning the motion of the vortex are illustrated. Finally, it is shown that an alternative interpretation of Magnus force [7] arises naturally in terms of the Hall equation. In Section 3 we incorporate the effect of dissipation and that of an external electric current in our field theoretical formalism and we derive the equation of motion for the vortex through explicit calculations. The results of an extensive numerical study are presented in Section 4. Vortices are found to behave in accordance to our earlier theoretical predictions. As a byproduct of this study we investigate the detail of the internal “Cyclotron motion” of vortices and we show that an isolated vortex is spontaneously pinned. In the concluding Section 5 we discuss the relevance of our results

to the actual superconductor and we propose more realistic 3-dimensional studies.

2 The Model

Most of our techniques and conventions are described elsewhere [1],[4], so we briefly outline here the physical and mathematical background of the theory. The model in question was originally introduced by Feynman [2] as a natural dynamical extension of the Ginzburg-Landau static theory of superconductivity. Indeed, by attributing the correct physical content to fields and parameters, it becomes a rather realistic phenomenological model of a superconductor [4]. The model admits infinitely long, smooth, cylindrically symmetric flux vortex solutions, whose static properties together with the properties of pairs of them, have been studied in detail in reference [3]. Our objective is to study some aspects of vortex dynamics within this model. In doing so, we will ignore excitations along the axis of the vortex (which by convention is taken to be parallel to z-axis) and we will directly define the model in two space dimensions.

As usual, the important dynamical variable is a complex order parameter Ψ which may be thought as an electrically charged field coupled to the electromagnetic potential (A_0, \mathbf{A}). The fields satisfy the coupled system of TDGL equations (to simplify notation, fields and coordinates are rescaled along the lines of reference [4])

$$\begin{aligned} i\dot{\Psi} &= -\frac{1}{2}\mathbf{D}^2\Psi + A_0\Psi + \frac{1}{4}\kappa^2(\Psi^*\Psi - 1)\Psi \\ \frac{1}{\beta}\dot{E}_i &= \epsilon_{ij}\partial_jB - J_i^s \quad \frac{1}{\beta}\partial_iE_i = \rho \end{aligned} \tag{2.1}$$

with $B = \epsilon_{ij}\partial_iA_j$, $E_i = -\partial_iA_i - \partial_iA_0$, $D_i = \partial_i - iA_i$, β, κ are free parameters, and the spatial indices i, j range from 1 to 2. The supercurrent density \mathbf{J}^s and the charge density ρ are given by $J_i^s = \frac{1}{2i}[\Psi^*D_i\Psi - c.c]$ and $\rho = \Psi^*\Psi - 1$. Note that in order to allow the possibility of a condensate ($|\Psi| = 1$) at infinity we introduce a background (positive-ion) charge density ρ_b to neutralize the system. For simplicity ρ_b is taken to be constant and homogeneous throughout i.e. $\rho_b = -\rho_s$ where ρ_s is defined as $\rho_s \equiv \Psi^*\Psi|_\infty = 1$.

The nonlinear system of equations (2.1) admits static, axially symmetric localized vortex solutions. They are similar in nature to the well known Abrikosov vortices of the static GL theory. Moreover, in a certain region of the parameter space their characteristic lengths (penetration depth, coherence length) fall well within the scale of a typical type II

superconductor [4]. However, the vortices in this model differ from the ordinary Abrikosov vortices mainly in one respect; although they carry zero electric charge as a whole, there is a local charge modulation in their interior i.e. they have non-vanishing charge density and as a result an electric field is associated with them.

A remarkable feature of the model in hand, is the unusual response of the vortices to external probes. Indeed, as it was shown in references [1],[3], the motion of the vortex is governed by the equation

$$V_L i = -\frac{1}{2\pi N} \epsilon_{ij} F_j \quad (2.2)$$

where \mathbf{V}_L is the mean velocity of the vortex and \mathbf{F} is the sum of the external forces acting upon it. N is the well-known integer-valued winding number or topological charge characterizing any finite energy configuration which counts the number of times the phase of Ψ rotates around the internal circle as we scan the circle at spatial infinity [8]. N is a conserved quantity and can be written as the integral of a properly chosen “topological density” τ . Among other possibilities τ may be defined as $\tau = \frac{1}{2\pi} B$, a definition which entails the familiar magnetic flux quantization and it is of obvious physical interest while, for our purposes the most useful form of τ is

$$\tau = \frac{1}{2\pi i} [\epsilon_{kl} (D_k \Psi)^* (D_l \Psi) - iB (\Psi^* \Psi - 1)] \quad (2.3)$$

The importance of the above formula stems from the fact that it appears in the expressions for the momentum and the angular momentum of the theory. Indeed, it was pointed out in reference [1], that the Noether expression for the linear momentum of the model is ambiguous for any configuration with non-zero topological charge. The unambiguous expression turns out to be

$$P_k = \epsilon_{ki} \int d^2x (2\pi x_i \tau + \frac{1}{\beta} E_i B) \quad (2.4)$$

which in turn leads to a radical revision of the physical interpretation of the momentum. The presence of the first moment of the topological density in the expression for the linear momentum, directly associates the latter with the position of the vortex. In fact, it is possible to show [1] that the “mean position” of the vortex is described by a quantity \mathbf{R} called the guiding center of the configuration and defined as $R_i \equiv -\frac{1}{2\pi N} \epsilon_{ij} P_j$. By identifying \mathbf{R} with the position and $\dot{\mathbf{R}}$ with the mean velocity \mathbf{V}_L of the vortex, and assuming that a generic force $F_j = \frac{dP_j}{dt}$ acts on the system, we end up with equation (2.2).

Contrary to intuition and Newtonian Mechanics, equation (2.2) implies that a vortex moves at a constant calculable speed in a direction perpendicular to the applied force.

This kind of behavior is analogous to the planar motion of charge particles under the action of a perpendicular magnetic field [9] and thus we call it Hall behavior and from now on we will refer to eq.(2.2) as Hall equation. Vortex dynamical behavior becomes less exotic and Newtonian Mechanics reestablishes by adopting an alternate interpretation of the Hall equation. To do so, we rewrite eq.(2.2) in the form : $\mathbf{V}_L = -\frac{1}{2\pi N} \mathbf{F} \times \hat{e}_z$ where \mathbf{V}_L is the velocity of the vortex in the $x - y$ plane and \mathbf{F} is the total force acting upon it. Multiplying both sides of the former equation by $2\pi N$ and taking the cross product by \hat{e}_z we get

$$2\pi N \mathbf{V}_L \times \hat{e}_z = \mathbf{F} \quad (2.5)$$

According to eq.(2.5), for a vortex which moves with constant velocity Newton's Law $\sum \mathbf{F} = 0$ is restored if we assume that in addition to any other force acting on the vortex another 'new' transverse force $\mathbf{F}_T = -2\pi N \mathbf{V}_L \times \hat{e}_z$ also acts on it. Bearing in mind that the total magnetic flux ϕ_0 equals $2\pi N$ we see that \mathbf{F}_T has similar form to the familiar so called 'Magnus force' [7] which is usually invoked to describe the motion of vortices in the superconductor. The similarity becomes more obvious in full units where this additional force reads $\mathbf{F}_T = -\rho_s \phi_0 \mathbf{V}_L \times \hat{e}_z$.

As we have explained in some previous work [3], Hall behavior (and consequently the transverse force \mathbf{F}_T) is a generic characteristic of soliton dynamics in systems with non trivial topology and spontaneously broken Galilean invariance, due for instance to the presence of a crystal lattice, and has a clear mathematical origin. Yet one would like to have a more physical explanation for the appearance of \mathbf{F}_T in eq. (2.5). In fact, one can attain such an explanation by attributing the origin of \mathbf{F}_T to the interaction between the magnetic flux of the vortex and the internal electric currents which are generated by the motion of the vortex [10]. Specifically, let us assume that the vortex is moving with constant velocity \mathbf{V}_L . In our field theory prescription a moving vortex with velocity \mathbf{V}_L is a field configuration $\Psi(\mathbf{r} - \mathbf{V}_L t) e^{i(\mathbf{V}_L \mathbf{r} - \frac{1}{2} |\mathbf{V}_L|^2 t)}, \mathbf{A}(\mathbf{r} - \mathbf{V}_L t)$ (see ref. [11] for more details on Galilean boosts in 2-d vortices). Let us now switch to the reference frame where the vortex is still. In that frame, the background ions of the crystal lattice form a homogeneous current of negative charge carriers with charge density $-\rho_s$ and velocity $-\mathbf{V}_L$. This current interacts with the magnetic field of the vortex and as a result feels a Lorentz force acting upon it. Consequently the vortex feels a backreaction force opposite to the Lorentz force which can be easily computed and is found to be exactly the transverse force \mathbf{F}_T mentioned above.

3 External Current and Dissipation

To study the response of the vortex to an externally prescribed current $\mathbf{J}^{ext}(\mathbf{x}, t)$ we simply substitute $\mathbf{J}^s \rightarrow \mathbf{J}^s + \mathbf{J}^{ext}$ in eq. (2.1). Because of the external current the linear momentum (2.4) of the system is no longer conserved. A straightforward application of the equations of motion yields

$$\frac{d}{dt} P_k = F_k^{Lorentz} = - \int d^2x \epsilon_{kl} J_l^{ext}(\vec{x}, t) B(\vec{x}, t) \quad (3.1)$$

Assuming that the external current is uniform throughout the plane i.e. $\vec{J}^{ext}(\vec{x}, t) = \vec{J}_0$, the only space dependent quantity on the right hand side of this equation is the magnetic field, and its integral, the total magnetic flux, is equal to $2\pi N$. Thus, the equation of motion for the momentum takes the simpler form $\dot{P}_k = -2\pi N \epsilon_{kl} J_{0l}$. Correspondingly the time evolution of the guiding center or the “mean position” of the vortex reads

$$\frac{dR_k}{dt} = -\frac{1}{2\pi N} \epsilon_{ij} \dot{P}_j = -J_{0k} \quad (3.2)$$

which quite surprisingly implies that the vortex drifts against the current with constant calculable speed. Note that in a similar model, Manton [11] arrives to the same result by considering Galilean boosts on vortex configurations.

The pull of the vortex by the current has a simple explanation in the context of Hall equation. The magnetic flux of the vortex exerts on the electric current a Lorentz force in the -90° direction with respect to the current. Consequently the vortex feels a back-reaction force in the $+90^\circ$ direction and naively one would expect the vortex to move in a direction perpendicular to the current. However, as it follows from the Hall equation the vortex moves at $+90^\circ$ with respect to the applied force, and therefore, the vortex moves in a direction opposite to the external current.

The introduction of dissipation in the system is a complicated task. Formally the effect of dissipation in the system is studied by adding a, phenomenological friction term in the TDGL equations. However, no such term has been derived on the basis of solid physical reasoning. Yet, one must have in mind that there are several restrictions in the form and the properties of any such term. Any friction term inserted in (2.1) should meet the following conditions : a) it should vanish for any static vortex solution, b) it should decrease the total energy W of the system i.e. $-dW/dt$ should be positive definite and finally c) should preserve the electromagnetic $U(1)$ gauge invariance of the system, or equivalently it should preserve the continuity equation. Our choice, though not unique

was the most natural among a small set of candidates, and that was to add a term of the form $C_d \epsilon_{ij} \partial_j \dot{B}$ -with C_d a positive constant- on the right hand side of the equation of motion for the electric field in (2.1).

By construction the friction term is gauge invariant and vanishes for any static configuration. The time derivative of the energy reads $dW/dt = \frac{1}{\beta} \int d^2x \{ E_i (C_d \epsilon_{ij} \partial_j \dot{B}) \}$ which by integration by parts becomes $dW/dt = -\frac{C_d}{\beta} \int d^2x \{ (\epsilon_{ij} \partial_j E_i) \dot{B} \} = -\frac{C_d}{\beta} \int d^2x \dot{B}^2$. Thus we conclude that the friction term we propose meets all the restrictions mentioned above and therefore it seems to be a reasonable candidate for a phenomenological study of dissipation.

The time evolution of the linear momentum is

$$\frac{d}{dt} P_k = C_d \int d^2x \{ \epsilon_{kl} \epsilon_{lm} \partial_m \dot{B}(\vec{x}, t) B(\vec{x}, t) \} = C_d \int d^2x \{ \dot{B}(\vec{x}, t) \partial_k B(\vec{x}, t) \} \quad (3.3)$$

To proceed further we adopt the rather plausible assumption that in the presence of a suitably chosen external current as well as dissipation a steady state is eventually reached in which the vortex moves rigidly with constant velocity \vec{V}_L . We assume further that the profile of the magnetic field associated with the steady state of the vortex is of the form $B(\vec{x}, t) = B(\vec{x} - \vec{V}_L t; \vec{V}_L)$. Then $\dot{B}(\vec{x}, t) = -V_{Lj} \partial_j B$ and eq. (3.3) reduces to

$$\frac{d}{dt} P_k = -S_{kj} V_{Lj} \quad \text{with} \quad S_{kj} = C_d \int d^2x \{ \partial_k B \partial_j B \} \quad (3.4)$$

A simpler version of this relation is obtained by invoking some further assumptions about the steady state profile of the vortex, which may be viewed as reasonable approximations in the limit of a weak external current. Thus, we assume that the vortex retains approximately its initial shape and correspondingly its axial symmetry i.e. in the rest frame of the vortex the magnetic field is of the form $B = B(\rho)$, where ρ is the radial coordinate in the usual polar variables. Then the diagonal terms of the S_{kj} tensor vanish while $S_{11} = S_{22} = C_d/2 \int d^2x (\partial_\rho B^2) \equiv \eta$ and equation (3.4) reduces to

$$\frac{d}{dt} P_k = -\eta V_{Lk} \quad (3.5)$$

with η being a positive constant number. Eq. (3.5) implies that the effect of such a term on an axially symmetric vortex configuration moving with velocity \mathbf{V}_L is a force linear to the velocity of the vortex which opposes the motion of the latter. We thus believe that with the insertion of this term we correctly incorporate the effect of friction in our model. Finally, the equation of motion of the guiding center of the vortex in the presence of both a homogeneous external current and dissipation reads

$$V_{Lk} \equiv \frac{dR_k}{dt} = -J_{0k} + \frac{1}{2\pi N} \epsilon_{km} \eta V_{Lm} \quad (3.6)$$

Equation (3.6) implies that a vortex in the presence of a transport current, simply drifts against it ($\mathbf{V}_L = -\mathbf{J}_0$) ; this leads to a perfect Hall effect but with opposite sign to that of the normal-state [12]. In the presence of dissipation the velocity of the vortex acquires a component in the perpendicular to the applied current direction. However, its longitudinal part still has opposite direction to the transport current, which in turn results in a sign change of the Hall effect [12]. By plain implementation of eq. (3.6) we find that the deflection angle δ between \mathbf{V}_L and \mathbf{J}_0 decreases from 180° to 90° as the contribution of the drag force increases. Furthermore, a simple relation for the deflection angle δ is obtained, namely

$$\cos\delta = -\frac{|\mathbf{V}_L|}{|\mathbf{J}_0|} \quad (3.7)$$

The former relation becomes obvious by simple inspection of fig.1. There are two forces acting upon the vortex ; i) the Lorentz force ($F_k^{Lorentz} = 2\pi N \epsilon_{kl} J_{0l}$) and ii) the dissipation force ($\mathbf{F}_d = -\eta \mathbf{V}_L$). According to eq. (2.2) the vector sum of \mathbf{F}_d and $\mathbf{F}_{Lorentz}$ rotated by $+90^\circ$ and divided by $2\pi N$ must be equal to the velocity of the vortex \mathbf{V}_L . The latter condition leads to relation (3.7).

From our analysis we conclude that within our model, and with or without dissipation the longitudinal part of the velocity of the vortex always has a direction opposite to the transport current which in turn results that the Hall effect in the vortex state will have a sign which is opposite to that of the normal state. We should note here that equation (3.6) is not original at all. The major contribution of this work is its field theoretical derivation and its interpretation. Indeed, by multiplying both sides of (3.6) by $2\pi N$ and taking the cross product with \hat{e}_z we can write the latter in the form

$$2\pi N \mathbf{V}_L \times \hat{e}_z = -2\pi N \mathbf{J}_0 \times \hat{e}_z - \eta \mathbf{V}_L \quad (3.8)$$

All the terms above appear in the most common phenomenological theories of vortex motion [13]. The left hand side of (3.8) is the familiar though controversial Magnus force while the first term of the right hand side is the so-called Lorentz force [14], or according to some other authors [15] an inseparable part of the Magnus force. Finally, the last term of the right hand side of (3.8) is like the viscous drag force of the Bardeen-Stephen (BS) model [13]. However, one should notice that the Lorentz force comes with opposite sign in eq. (3.8) with respect to what is commonly accepted in the literature, a difference which is of critical importance. In fact, the sign inversion of the longitudinal part of the vortex velocity in our analysis, has its origin in the sign change of the Lorentz force. The reason for the latter change is in the way we incorporate the external current in our model. We think of the current as a totally substantive object which interacts with the vortex. It is

like a vortex moving in a plane under the influence of a current which flows in a parallel plane just above the vortex plane, instead of a current flowing in the plane of the vortex and formed by the same carriers as those of the vortex [13]. At first sight the second approach looks more natural. However, if we think of the vortex as a 3-dim object i.e. as a flux tube formed in a 3-dim superconducting film and interacting with supercurrent which is formed only on the top and the bottom of the film, our approach becomes more plausible and realistic.

4 Numerical Results

Our next objective is the numerical investigation of the dynamical behavior of the vortices in the presence of an external electric current and dissipation. Our computational techniques are described in detail elsewhere [4], so here we present only a brief overview.

To simulate the motion of a vortex, we first determined numerically the static profiles of the condensate and gauge fields characterizing a vortex of winding number N [3]. With these in hand, we laid down on the lattice a configuration of a vortex centered at $(x, y) = (2, 0)$ at time $t = 0$. In order to maintain (as much as possible) the symmetries of the continuous system in its discretized form, we have resorted to techniques from lattice gauge theory [4], [16]. The degrees of freedom were discretized on a spatial lattice so as to maintain exact (lattice) gauge invariance. The time evolution was implemented by a finite difference leapfrog method using equations (2.1) with the second one modified to

$$\frac{1}{\beta} \dot{E}_i = \epsilon_{ij} \partial_j B - J_i^s - J_i^{ext} + C_d \epsilon_{ij} \partial_j \dot{B} \quad (4.1)$$

in order to incorporate the effect of an external current and dissipation. The gauge freedom of the equations of motion was eliminated by imposing the temporal gauge $A_0 = 0$. The external current \mathbf{J}^{ext} was taken along the x -direction and uniform throughout the whole plane i.e. $\mathbf{J}^{ext} = J_0 \hat{\mathbf{e}}_x$.

In any numerical calculation where partial derivatives are involved, the imposition of appropriate boundary conditions is a very delicate task. Here, the presence of an incoming and an outgoing external current at $\pm x$ -infinity, made this issue even more complex and forced us to use different boundary conditions at x and y -boundaries. At the y -boundaries of the film we imposed free boundary conditions by setting the covariant derivative in the normal to the boundary direction equal to zero ($D_y \Psi = 0$). There are more than one ways to impose such a condition, and our choice was to set $\partial_y \Psi = 0$ and $A_y = 0$. Also, in order to get vanishing magnetic field at the y -boundaries we set

$\partial_y A_x = 0$. At the x -boundaries we successfully imposed two different sets of b.c. There, in deriving the boundary conditions we took special care in order to preserve the discrete gauge invariance of the system, or equivalently the discrete version of the continuity equation, at the x -boundaries. Specifically, in analogy to the y -boundaries we set $\partial_x A_y = 0$, $A_x = 0$ and then, by imposing the continuity constraint we got $\frac{1}{i}[\Psi^* \partial_x \Psi] = -J_0$. As a consequence of the above b.c. the value of the x-component of the supercurrent at $\pm x$ -infinity was fixed and equal to $-J_0$. Even though, this is what we expect to happen away from the vortex, still the b.c. for Ψ sounds quite artificial and too constrained. Thus, we tried and finally used another more “natural” set of b.c. Namely we set $\partial_x A_x = \partial_x A_y = 0$ and the constraint of preserving the continuity equation yielded $D_x^2 \Psi = 0$ as a b.c. for Ψ . Using both sets we got essentially identical results, and these have been exhaustively checked to be free of any boundary contributions.

The simulations were done on a square lattice of 201×201 sites with a lattice spacing $\alpha = 0.15$. The width (diameter) of the vortex was typically of the order of 2 in rescaled length units. The finite time step was chosen to be much smaller than the lattice spacing typically of the order 10^{-3} . To test our results we ran simulations in bigger lattices 401×401 with the same or smaller lattice spacing, say $\alpha = 0.1$, and the results obtained were all perfectly consistent. All our simulations were performed on various HP workstations at the University of Crete. A typical run of duration $T \approx 100$ time units, with $\Delta t = 0.002$ on a 201×201 lattice needed about 15 hours of CPU time on a HP-735 machine.

Apart from the existence of the external current and the new boundary conditions, the algorithm we used here was identical to the one previously used in the numerical study of a vortex pair dynamics [4]. There, it turned out that the algorithm was extremely accurate. Here, as a sort of calibration of the algorithm, and mainly to avoid any systematic contributions from the b.c. we initially tested it in a system where the result is known analytically, namely at the trivial sector $N = 0$. There, it is easy to see that the field configuration $\Psi(\mathbf{x}, t) = \exp[-\frac{i}{2} J_0^2 t]$, $A_x(\mathbf{x}, t) = J_0$, $A_y(\mathbf{x}, t) = 0 = A_0(\mathbf{x}, t)$ is a solution of the equations of motion in the presence of the external current with $\mathbf{E}(\mathbf{x}, t) = 0 = \mathbf{B}(\mathbf{x}, t)$ and $\mathbf{J}^s(\mathbf{x}, t) = -\mathbf{J}^{ext}$. We ran a preliminary simulation using as initial configuration the trivial vacuum $\Psi = 1$, $A_i = 0 = A_0$ and turning on the external current at $t = 0$. Our naive expectation was to see the system relaxing to the vacuum-current solution described above or to some gauge transform of it. Yet, after a short transient period the system dynamically relaxed to a time-dependent configuration where both the electric field and the supercurrent oscillated vividly around their mean values $\langle \mathbf{E}(\mathbf{x}) \rangle = 0$, $\langle \mathbf{J}^s(\mathbf{x}) \rangle = -\mathbf{J}^{ext}$. These oscillating modes could be attributed to the abrupt turning on of the external current.

To eliminate them, we introduced dissipation in the system and in a subsequent run we saw the system relaxing to the vacuum-current solution within a 10^{-4} accuracy. (Note that we introduced here a dissipation term of the form $\dot{E}_i = -C'_d E_i + \dots$ which does not meet all the criteria mentioned in the previous section and thus it is not appropriate in the vortex sector). The remarkably accurate convergence of the initial configuration to the vacuum-current solution provides a strong confirmation for the accuracy and the reliability of our algorithm. A byproduct of these runs is the conclusion that the response of the ground state to the application of an external electric current is the formation of a supercurrent which in average is equal and opposite to the latter. In fact, it is reasonable to assume that even in the $N \neq 0$ sector, away from the vortex, the system will respond in the same way.

We switch now to the study of the vortex sector and we particularly consider the $N = 1$ sector. To study the dynamics of the vortices we carried out numerous simulations and the results confirmed with quite impressive accuracy the predictions of the theoretical analysis. We also experimented with the values of the parameters κ , β and the dynamics of the vortices showed little sensitivity to those values. Indicatively we quote here the results of some simulations for the specific choice $\kappa = 2$ and $\beta = 1$, which belong to a parameter regime that we believe to be appropriate for the description of the physics of type II superconductors [4]. Similar results though, were obtained for a large variety of the values of the parameters.

We performed a series of numerical experiments for various values of the dissipation constant C_d and for a fixed value of the external current $J_0 = 0.025$. The total duration of each simulation was 240 time units. The corresponding trajectories of the guiding center of the vortex are displayed in fig.2. As arises from the plot, in the absence of dissipation ($C_d = 0$), the guiding center of the vortex performs a rectilinear trajectory along the negative x-direction. Moreover, it is displaced by 6 space units (i.e. three times its diameter) in 240 time units, which corresponds to a mean velocity $\mathbf{V}_L = -J_0 \hat{e}_x$. This behavior verifies the theoretical prediction (3.2) qualitatively as well as quantitatively.

To arrive to eq. (3.2) we assumed that the external current is homogeneous throughout the whole space. This assumption may take a weaker form; instead of a current occupying the whole space, we may introduce one which has a strip shape, i.e. it takes a non-zero constant value inside a strip of certain width, while it vanishes in the outside region. In principle, when the width of the strip is much larger than the size of the vortex, the vortex essentially realizes a homogeneous current all over space and responds accordingly. This assumption was tested and verified in our study. Specifically, we repeated our runs for

$C_d = 0$ using an external current of the form

$$J_y^{ext} = 0 \quad J_x^{ext} = J_0 f(y) \quad (4.2)$$

where the function f is approximately equal to unity over a strip of width $2L$ and drops to zero very quickly outside that strip. A function which meets the above description is

$$f(y) = e^{-(|y|/L)^n} \quad (4.3)$$

especially for large n . The numerical calculations showed that any strip current with $L \geq 6$ has the same effect on the motion of the vortex to that of a homogeneous current occupying the whole space. Quite surprisingly, even for narrower strips, $L \leq 6$ the trajectory of the vortex remains identical while its speed reduces. Indicatively, we found the ratio $-\frac{V_x}{J_0}$ = 0.89, 0.95 and 0.98 for $L = 3, 4$ and 5 respectively.

For non-zero values of C_d , after a small transient period the vortex relaxes to an almost rectilinear motion at an angle to the external current different from 180° . Indeed, the deflection angle δ between the trajectory of the vortex and the direction of the current takes values from 180° to 125° with δ decreasing as C_d increases. As arises from the lengths of the trajectories in fig.2, the measure of the velocity of the vortex is also a decreasing function of C_d in accordance to the physical interpretation of dissipation.

An important issue to check is to which extent the motion of the vortex meets the assumptions we adopted in section III while deriving eq. (3.7). To see whether a steady state is eventually reached, we plot in fig.3 the time evolution of the x and y -components of the drift velocity of the vortex for $C_d = 4$. There, we see that initially ($t = 0$), $V_x = -0.025 = -J_0$ and $V_y = 0$ as if there was no dissipation at all. As the vortex moves on, dissipation turns on and its effect results in a non-zero transverse component of the vortex velocity. After a small transient period ($t = 0 - 4$), the velocity of the vortex sets in an oscillating mode around a constant mean value. Strictly speaking the vortex does not seem to develop a steady state, but it is reasonable to assume that the contribution of the oscillating part is auto-canceled in average, and thus in a wider sense we can assert that the vortex finally relaxes to a steady state. Also, a detailed examination of successive level contours of the energy density establishes that the vortex moves quite coherently and retains its initial shape during the evolution of the simulations. It is thus quite interesting to question whether the trajectories shown in fig.2 satisfy eq. (3.7). To calculate the deflection angle and the (mean) velocity \mathbf{V}_L of the vortex we process the numerical data so as to linearize the slightly wavy trajectories of fig.2 and we cut out the initial part of the data which corresponds to the “transient period”. After the relevant

calculations we tabulate the results in table I, where it is demonstrated that relation (3.7) is satisfied with quite impressive accuracy.

Table I

C_d	$-cos\delta$	$\frac{V_L}{J_0}$	$-\frac{cos\delta}{V_L/J_0}$
0.	1.000	1.000	1.000
1.	0.984	0.981	1.003
2.	0.942	0.934	1.009
4.	0.816	0.803	1.016
8.	0.579	0.582	0.995

It is of crucial importance for the present study to determine to which extent the details of the motion of the vortex follow the motion of its guiding center. To investigate the latter we must somehow define the position of an extended object such as the vortex. Thinking of the vortex as an energy lump and given that this energy lump moves coherently retaining its initial axially symmetric shape, we believe that the maximum of the energy density (MED) of the vortex configuration accurately describes the real position of the vortex. Therefore, in our calculations we also keep a record of the location of the MED of the vortex configuration as time evolves. Note that there are at least two alternative definitions of the position or the center of the vortex, namely the maximum of the topological density or the position where Ψ vanishes. All three of them yield similar results so here we present only data for the MED. The trajectories of the MED of the vortex (solid line) and of the guiding center (dashed line) of the vortex for $C_d = 0$ and for various values of the intensity J_0 of the external current are plotted in fig.4. We see that while the guiding center simply drifts in the negative x-direction, the motion of the MED is more complicated. In average it follows the motion of the guiding center, but its trajectory is modulated by an oscillatory pattern. This modulation is not a numerical effect but is an inherent characteristic of vortex dynamics. We have already encounter similar oscillating patterns in an earlier work while studying the dynamical evolution of a pair of vortices [4]. These patterns are reminiscent of the motion that electrically charged particles perform in the presence of a magnetic field, which happens to be the prototype physical system which exhibits Hall behavior. Borrowing the terminology from the latter, we will refer to this finer motion with the name “cyclotron”. As in the original case, the amplitude of the cyclotron motion varies with the parameters of the problem. Along with the amplitude, the importance of cyclotron motion also varies. In the extreme limit where the amplitude of the cyclotron motion is very large in comparison with the length scale of the problem we study, the whole picture of the dynamical behavior of vortices, as this is

determined from the equation of motion for the guiding center alters dramatically. Thus, it is important to determine the way the parameters of the model affect cyclotron motion. Our numerical investigation revealed a systematic relation between the parameter β , the value of the external current J_0 , and the amplitude of the oscillating patterns. In short, the cyclotron motion is amplified when β decreases as well as when J_0 increases. The dependence on J_0 and β is demonstrated in fig.4 and fig.5 respectively. Note that in the runs presented in fig.4 $\kappa = 2$. and $\beta = 1$. while in those in fig.5 $\kappa = 1.5$ and $J_0 = 0.03$. Here we stress once more the analogy between the motion of the vortex in this model and the Hall motion of charged particles. The increase of J_0 here is equivalent to the increase of the external electric field in the charged particle system and the effect on the cyclotron motion is the same in the two cases. Also the increase of β is physically equivalent to the increase of the dielectric constant. The analogue in the particle system is to decrease the coupling of the particle with the electric field, without affecting its coupling with the magnetic field. Again the consequence is similar in the two systems.

Finally we consider the case where an initially applied external field is abruptly turned off. The response of the vortex to such a “blackout” is of obvious interest. Hall equation (2.2) implies that in the absence of external forces the guiding center of the vortex is conserved i.e. the vortex is pinned. In other words, while the vortex moves at a constant speed just before the current is turned off, it abruptly freezes at the position where it is found at the time we switch off the current. Still there is one question to be answered, namely, how the location of the MED -which in principle does not coincide with the guiding center- will evolve. After all the guiding center is an abstract notion which represents the “mean position” of the vortex, while its real position in space is associated with the distribution of the energy density. One could possibly assume that after the pause of the current the vortex will reorganize itself and will finally relax to a configuration where the MED coincides with the pinned guiding center. However, the numerical results lead to a completely different picture.

In fig.6 (a), (b) we display the results of a simulation of a total duration of 125 time units where an external current is on only up to time $t = t_{crit} = 40$. The trajectories of both the guiding center (solid line) and the MED (dashed line) are plotted. Plot (a) shows the trajectories from $t = 0$ up to $t = t_{crit}$ while plot (b) displays the trajectories till $t = 125$. As it is shown there, after the current is turned off, the guiding center indeed remains fixed at the point where it was found then, while the maximum of the energy density sets in a circular motion around the location of the guiding center with a radius equal to its distance from the guiding center at the time the current was turned off.

The picture described here, has a striking analogy with the Hall effect. A guiding center can be also introduced in the case of the two-dimensional electron motion in a uniform magnetic field B [9] which again can be interpreted as the “mean position” of the electron. When an external electric field is applied the electron sets in a cycloid motion along the Hall direction while its guiding center follows a rectilinear orbit along the same direction. What is more, when the electric field is turned off, the electron, sets in a circular motion around its guiding center which rests.

5 Discussion

In this paper, a time-dependent Ginzburg-Landau model for a complex scalar field coupled to electromagnetism has been studied mainly numerically. Dissipation was successfully incorporated in the model and its effect on the motion of the vortex was analytically determined. The results of the numerical study were in accord to the Hall analogy advocated in ref. [1] based on the derived unambiguous conservation laws. Furthermore, the results confirmed with impressive accuracy earlier theoretical predictions concerning the speed and the direction of the velocity of the vortex. In short, it was shown that under the influence of an external electric current a vortex drifts in a direction opposite to the current while in the presence of dissipation it deflects in a direction ranging from 90° to 180° with respect to the current.

An important feature which is worth emphasizing is the cyclotron motion i.e. the oscillating patterns in the trajectory of the vortex remarkably similar to those encountered in the cycloid motion of an electron in the standard Hall effect. Such patterns have been already observed in the motion of vortex pairs [4] and in the motion of magnetic bubbles in ferromagnetic media [17]. It seems, that cyclotron motion is a generic feature of solitons which exhibit Hall behavior. According to reference [3] there is a whole class of field theories whose solitons are expected to exhibit Hall behavior and among them there are some interesting variations of the present model [11], [18]. In some sample runs in these models we did encounter cyclotron motion which we consider as a strong indication that indeed these systems exhibit Hall behavior. Note that in the framework of collective coordinate schemes like those invoked in [11], [19] it is not possible to detect cycloid patterns, because in the adiabatic limit the amplitude of cyclotron motion becomes negligible, a fact which is supported by the results displayed in fig.4.

At a phenomenological level, we have presented arguments (mainly by reformulating previous results), which are quite encouraging for the relevance of the model to the physics

of the superconductor. In particular, we have shown that Hall equation (2.2) leads naturally to the introduction of Magnus force (2.5) in the equation of motion for the vortex. Furthermore, we argued that the Magnus force has electromagnetic origin due to the interaction of the moving magnetic flux with the positive ions of the background lattice. Finally, we have derived a variation of the Nozieres-Vinen equation for the motion of a vortex (3.8) using plain field theoretical analysis.

We have also exhibited analytical considerations and numerical results, which suggest that in the presence of an external electric current the vortex drifts against the current implying a possible link with the opposite sign Hall effect [21]. Yet, as we mentioned in section IV, the way we incorporate the electric current in the model plays crucial role to the derivation of that result. Clearly the next step is to test the predictions of the model at hand against more realistic experimental situations. One should find a more natural way to introduce the electric current in the specimen. Also one should abandon the 2-dim reduction and study the issues presented here in thin films with finite thickness. In such a study [20], preliminary results imply that Hall behavior is also exhibited in the motion of magnetic flux tubes which are probed by a surface external current in a 3-dim film i.e. a current which is non-zero only at the upper and the lower layers of a 3-dim grid. At the experimental front, one might try to mobilize vortices not by applying an electric potential on the specimen, but by introducing an electric current in a parallel plane just above the specimen.

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FIGURE CAPTIONS

Figure 1. A plot of the forces acting upon a vortex which moves with constant velocity \mathbf{V}_L in the presence of a uniform external current \mathbf{J}_0 and dissipation.

Figure 2. The trajectory of the guiding center of the vortex under the influence of an external electric current and dissipation. The several lines correspond to different values of the friction parameter C_d .

Figure 3. The drift velocity $\mathbf{V} = (V_1, V_2)$ of the guiding center of the vortex for $C_d = 4$.

Figure 4. The motion of the vortex as determined by its guiding center (dashed line) and the location of the maximum of the energy (solid line) for various values of the external current.

Figure 5. The motion of the vortex as determined by its guiding center (dashed line) and the location of the maximum of the energy (solid line) for various values of the parameter β .

Figure 6. The pinning of the vortex. (a) the trajectory of the vortex while the current is on, and (b) the subsequent evolution after the current is turned off.

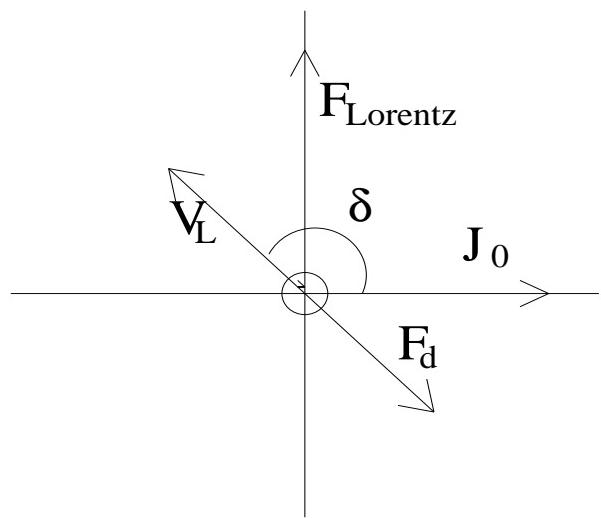


Figure 1

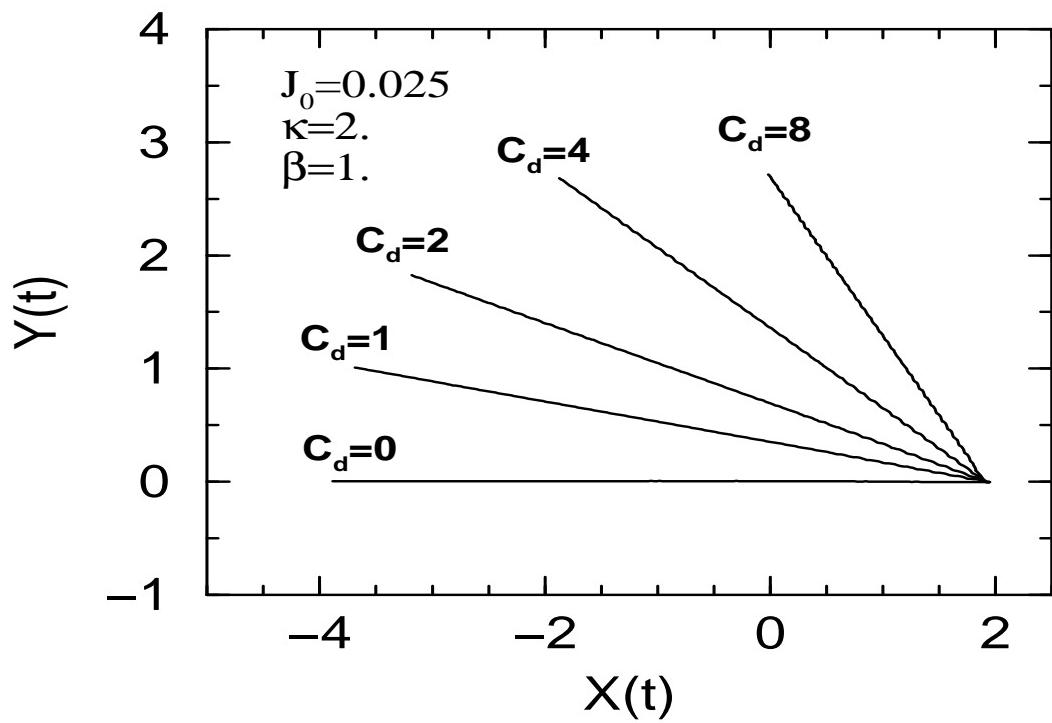


Figure 2

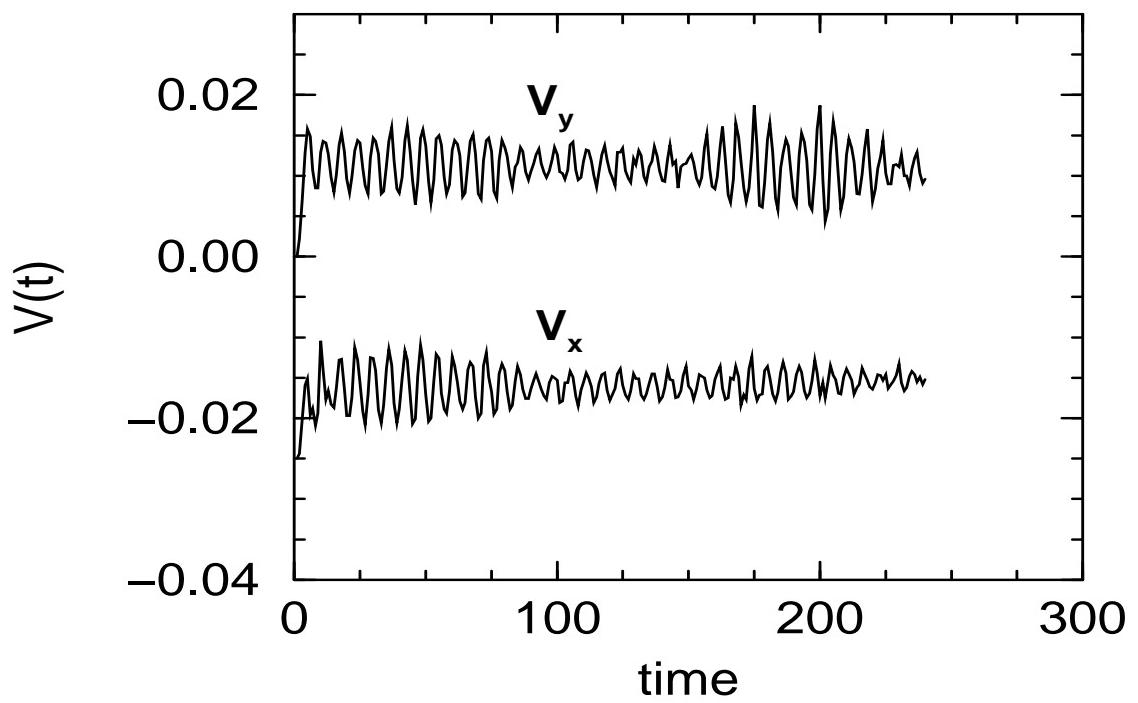


Figure 3

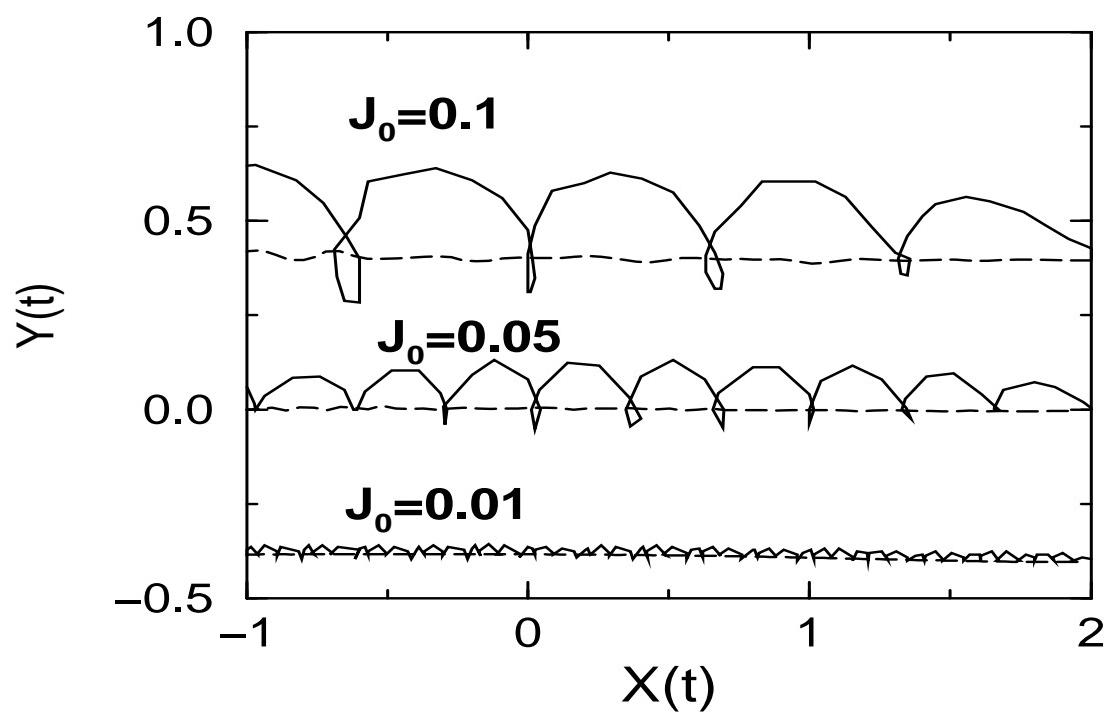


Figure 4

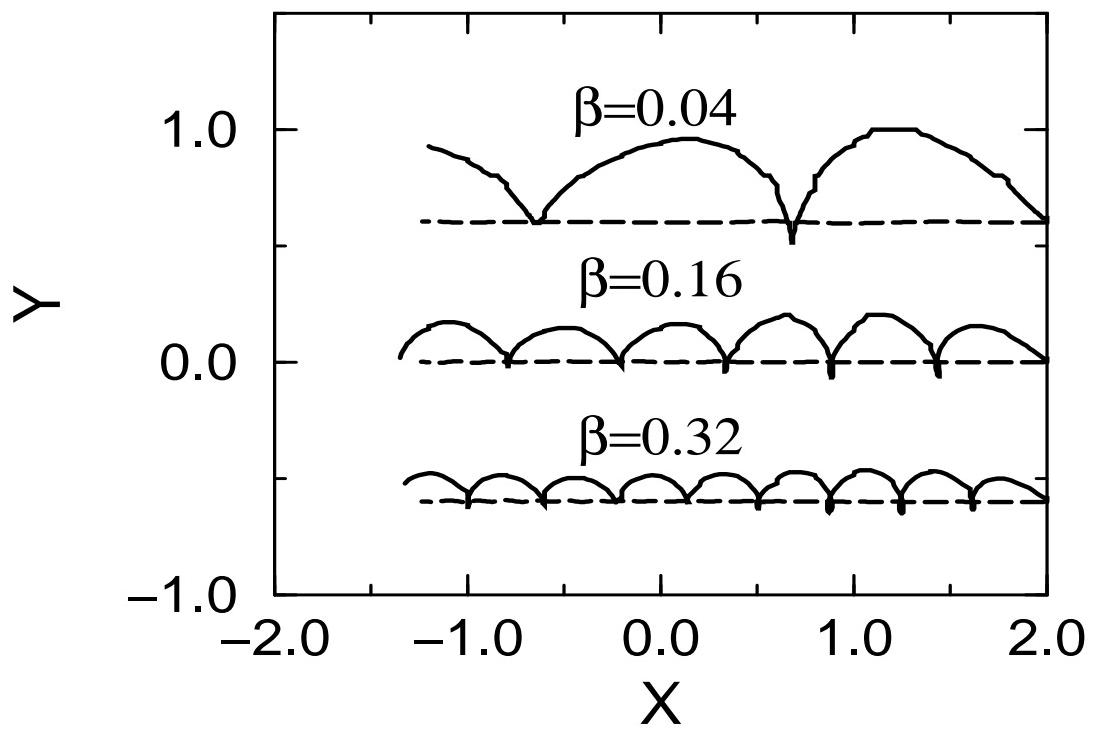


Figure 5

